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Closed-form solutions for the magnetoelectric coupling coefficients in fibrous composites with piezoelectric and piezomagnetic phases

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Abstract

This paper presents an analytical method to investigate the magnetoelectric coupling effect that is a new product property of piezoelectric–piezomagnetic intelligent composites since it is not present in each constituent. Based on the eigenstrain formulation and the Mori–Tanaka theory, the magneto–electro–elastic Eshelby tensors and the effective material properties of the composite are obtained explicitly. Particularly when both the matrix and the inclusions of the composite are transversely isotropic with different magneto–electro–elastic moduli, and shapes of inclusions are of elliptical cylinder, circular cylinder, disk, and ribbon, simple and closed-form solutions for the magnetoelectric coupling coefficients are acquired. The solutions are a function of the shape of inclusion, phase properties, and volume fraction of inclusions. Moreover, the derived simple expressions also show that the magnetoelectric coupling coefficients vanish as the volume fraction of inclusions tends to zero or one. This verifies that the magnetoelectric coupling coefficients are absent in each phase of the composite.

Keywords: Magnetoelectric coupling effect; Piezoelectric–piezomagnetic composites; Mori-Tanaka theory

1. Introduction

Combining two or more distinct piezoelectric and piezomagnetic (magnetostrictive) constituents, piezoelectric/piezomagnetic composite materials can take the advantages of each constituent and consequently have superior coupling magnetoelectric effect as compared to conventional piezoelectric or

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piezomagnetic materials. The magnetoelectric coupling effect is a new product property of the composite, since it is absent in each constituent. In some cases, the coupling effect of piezoelectric/piezomagnetic composites can be even obtained a hundred times larger than that in a single-phase magnetoelectric material. Consequently, they are extensively used as magnetic field probes, electric packaging, acoustic, hydrophones, medical ultrasonic imaging, sensors, and actuators with the responsibility of magneto–electro–mechanical energy conversion. These composites are referred to as *smart* or *intelligent* materials which can feedback the internal states of a material or structure. This may explain why piezoelectric/piezomagnetic composite materials constitute an important branch of the recently emerging technologies of modern intelligent materials. Particularly, applications in magnetic sensors could be very helpful for detecting both ac and dc magnetic fields. This suggests potential usage in magnetic storage and read-out devices, in magnetic imaging technology, and for shielding and protecting database by sensing and shielding from damaging magnetic fields Avellaneda and Harshe (1994).

The development of piezoelectric–piezomagnetic composites has its roots in the early work of van Suchtelen (1972) who proposed that the combination of piezoelectric–piezomagnetic phases may exhibit a new material property — the magnetoelectric coupling effect. Since then, the magnetoelectric coupling effect of $BaTiO_3$–$CoFe_2O_4$ composites has been measured by many researchers: van Run et al. (1974), van den Boomgaard et al. (1974) among others. They have shown that there indeed exists a remarlable magnetoelectric coupling effect in such piezoelectric–piezomagnetic composites. Much of the theoretical work for the investigation of magnetoelectric coupling effect has only recently been carried out by Harshe (1991), Harshe et al. (1993), Avellaneda and Harshe (1994), Nan (1994), Benveniste (1995) and Huang and Kuo (1997). It appears that these approaches have not provided a means to find closed-form solutions of the magnetoelectric coupling effect. Thus, the present work is an attempt to fill this information need.

In the present paper, a presentation of some notations used, the basic theory and equations on which the rest of this paper is built is in order in Section 2. In Section 3, the analytical solution for the coupled magneto–electro–elastic behavior of piezoelectric–piezomagnetic composites developed in the author's previous work (Huang and Kuo, 1997) is utilized to derive a set of nine tensors for an ellipsoidal inclusions in an infinite piezomagnetic matrix. These tensors will be referred to as the magneto–electro–elastic Eshelby tensors analogous to the Eshelby tensor in elasticity (Eshelby, 1957). In addition, closed-form expressions for the magneto–electro–elastic Eshelby tensors for some inclusions such as elliptical cylinder, circular cylinder, penny shape, and ribbon embedded in a transversely isotropic (6 mm symmetry) piezomagnetic medium are presented. These four inclusions are practically important in applications and are usually discussed at once by the micromechanics and composite communities. Section 4 takes the results derived in previous sections and applies them to determine the magnetoelectric coupling coefficients existing in piezoelectric–piezomagnetic composites. As a result, the magnetoelectric coupling coefficients are obtained in closed forms for the composite reinforced by elliptic cylindrical, circular cylindrical, penny-shape, and ribbon-like inclusions, respectively.

2. Some preliminaries

2.1. The inclusion problems

Before proceeding some notations used in this article are introduced. The usually summation convention applies to repeated subscripts with the exception that both lowercase and uppercase subscripts are used. Lowercase subscripts take on the range 1, 2, 3, while uppercase subscripts range from 1 to 5. Thus, $T_J U_J = T_j U_j + T_4 U_4 + T_5 U_5$, where $j = 1 \rightarrow 3$. With this shorthand notation, the magneto–electro–elastic moduli of a piezomagnetic material is conveniently expressed as

$$L^0_{iJMn} = \begin{cases} C^0_{ijmn} & J,\ M \le 3, \\ q^0_{nij} & J \le 3,\ M = 5, \\ q^0_{imn} & J = 5,\ M \le 3, \\ -\kappa^0_{in} & J = M = 4, \\ -\Gamma^0_{in} & J = M = 5, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Similarly the magneto–electro–elastic moduli of a piezoelectric material is represented as

$$L^1_{iJMn} = \begin{cases} C^1_{ijmn} & J,\ M \le 3, \\ e^1_{nij} & J \le 3,\ M = 4, \\ e^1_{imn} & J = 4,\ M \le 3, \\ -\kappa^1_{in} & J = 4,\ M = 4, \\ -\Gamma^1_{in} & J = 5,\ M = 5, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In the preceding equations, the superscripts '0' and '1' denote quantities in the matrix and the inhomogeneity respectively; C_{ijmn} denotes elastic moduli, e_{nij} is the piezoelectric coefficient, q_{nij} is the piezomagnetic coefficient, κ_{in} is the dielectric constant and Γ_{in} is the magnetic permeability. It is noted that, in general, the dielectric constant and magnetic permeability are neglected for piezoelectric and piezomagnetic materials, respectively. However, they are retained in the present work in order to investigate the magnetoelectric coupling effect in piezoelectric–piezomagnetic composites.

Now, consider an infinite piezomagnetic material containing an ellipsoidal inclusion whose material properties are the same as the matrix and is defined as

$$\Omega: \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \le 1, \tag{3}$$

where a_1, a_2 and a_3 are the lengths of the semiaxes of the ellipsoid. Let Z^*_{Mn} represent eigenstrain (or stress-free transformation strain ε^*_{mn} when $M \le 3$), eigenelectric field (or electric displacement-free electric field $-E^*_n$ when $M = 4$), and eigenmagnetic field (or magnetic induction-free magnetic field $-H^*_n$ when $M = 5$) in the inclusion, and zero in the matrix. The stress σ_{ij}, the electric displacement D_i, and the magnetic induction B_i, in the inclusion caused by a constant Z^*_{Mn} in Ω can be expressed as:

$$\sigma_{ij} = C^0_{ijmn}\left(\varepsilon_{mn} - \varepsilon^*_{mn}\right) - q^0_{nij}\left(H_n - H^*_n\right),$$

$$D_i = \kappa^0_{in}\left(E_n - E^*_n\right)$$

$$B_i = q^0_{imn}\left(\varepsilon_{mn} - \varepsilon^*_{mn}\right) + \Gamma^0_{in}\left(H_n - H^*_n\right), \tag{4}$$

in which ε_{mn} represents the elastic strain tensor, E_n is the electric field, and H_n is the magnetic field. In the shorthand notation, the constitutive equation Eq. (4) can be unifined into a single equation:

$$\Sigma_{iJ} = L^0_{iJMn}\left(Z_{Mn} - Z^*_{Mn}\right), \tag{5}$$

where

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & J \le 3, \\ D_i & J = 4, \\ B_i & J = 5, \end{cases}, \quad Z_{Mn} = \begin{cases} \varepsilon_{mn} = S_{mnAb} Z^*_{Ab} & M \le 3, \\ -E_n = S_{4nAb} Z^*_{Ab} & M = 4, \\ -H_n = S_{5nAb} Z^*_{Ab} & M = 5, \end{cases} \tag{6}$$

with S_{MnAb} being a collection of 9 tensors that are referred to as the magneto–electro–elastic Eshelby tensors analogous to the Eshelby tensor for elastic inclusion problems (Eshelby, 1957). As will be seen in the subsequent development, the magneto–electro–elastic Eshelby tensors are the key ingredients necessary for determining the magnetoelectric coupling of piezoelectric–piezomagnetic composites. It is useful to express S_{MnAb} explicitly in terms of the magneto–electro–elastic moduli L^0_{iJMn}, i.e.,

$$S_{mnab} = \frac{1}{8\pi}\left\{ C^{\,0}_{iJAb}(G_{mjin} + G_{njim}) + q^0_{iab}(G_{m5in} + G_{n5im}) \right\},$$

$$S_{mn4b} = \frac{-1}{8\pi}\kappa^0_{ib}(G_{m4in} + G_{n4im}),$$

$$S_{mn5b} = \frac{1}{8\pi}\left\{ q^0_{bij}(G_{mjin} + G_{njim}) - \Gamma^0_{ib}(G_{m5in} + G_{n5im}) \right\},$$

$$S_{4nab} = \frac{1}{4\pi}\left(C^{\,0}_{iJAb} G_{4jin} + q^0_{iab} G_{45in} \right),$$

$$S_{4n4b} = -\frac{1}{4\pi}\kappa^0_{ib} G_{44in},$$

$$S_{4n5b} = \frac{1}{4\pi}\left(q^0_{bij} G_{4jin} - \Gamma^0_{ib} G_{45in} \right),$$

$$S_{5nab} = \frac{1}{4\pi}\left(C^{\,0}_{iJAb} G_{5jin} + q^0_{iab} G_{55in} \right),$$

$$S_{5n4b} = -\frac{1}{4\pi}\kappa^0_{ib} G_{54in}$$

$$S_{5n5b} = \frac{1}{4\pi}\left(q^0_{bij} G_{5jin} - \Gamma^0_{ib} G_{55in} \right), \tag{7}$$

in which G_{MJin} is defined by (Mura, 1987; Huang and Kuo, 1997)

$$G_{MJin} = \int_{-1}^{1} \int_{0}^{2\pi} N_{MJ}(\bar{\xi}) D^{-1}(\bar{\xi}) \bar{\xi}_i \bar{\xi}_n \, \mathrm{d}\theta \, \mathrm{d}\bar{\xi}_3, \tag{8}$$

with $N_{MJ}(\bar{\xi})$ and $D(\bar{\xi})$ being the cofactor and the determinant of the 5×5 matrix, $[L^0_{iJMn}\bar{\xi}_i\bar{\xi}_n]$.

2.2. Effective magneto–electro–elastic moduli

Suppose that a sufficiently large two-phase composite consists of randomly oriented piezoelectric ellipsoidal inhomogeneities (Ω_1, Ω_2,..., Ω_N) with magneto–electro–elastic moduli L^1_{iJMn} and volume fraction f. The surrounding matrix is piezomagnetic and has a magneto–electro–elastic moduli L^0_{iJMn}. The effective magneto–electro–elastic moduli, $\bar{L}_{iJMn}$, of the composite have been obtained by Huang and Kuo (1997) through the Mori–Tanaka theory (Mori and Tanaka (1973)), incorporated with the equivalent inclusion method (Eshelby, 1957) as

$$\bar{L}_{iJMn} = L^0_{iJAb} + fL^0_{iJAb}V^{-1}_{AbqR}\left(L^1_{qRMn} - L^0_{qRMn}\right)\Big], \tag{9}$$

where V^{-1}_{AbiJ} is the inverse of V_{iJAb}, defined by

$$V_{iJAb} = (1-f)\left(L^1_{iJMn} - L^0_{iJMn}\right)S_{MnAb} + L^0_{iJAb}. \tag{10}$$

Due to the coupling interaction between magnetostrition of piezomagnetic phase and piezoelectricity of piezoelectric phase, the effective composite properties, $\bar{L}_{iJMn}$, should be comprised of the magnetoelectric coupling effect. Thus, the unified notation, $\bar{L}_{iJMn}$, in Eq. (9) is defined as:

$$\bar{L}_{iJMn} = \begin{cases} \bar{C}_{ijmn} & J,\ M\le 3, \\ \bar{e}_{nij} & J\le 3,\ M=4, \\ \bar{q}_{nij} & J\le 3,\ M=5, \\ \bar{e}_{imn} & J=4,\ M\le 3, \\ -\bar{\kappa}_{in} & J=4,\ M=4, \\ -\bar{\lambda}_{in} & J=4,\ M=5;\ J=5,\ M=4, \\ \bar{q}_{imn} & J=5,\ M\le 3, \\ -\bar{\Gamma}_{in} & J=5,\ M=5, \end{cases} \tag{11}$$

in which $\bar{\lambda}_{in}$ stands for the magnetoelectric coupling coefficient that is absent in the constituents of the composite. Thus, the overall stress, electric displacement, and magnetic induction in the composite are expressed as

$$\bar{\sigma}_{ij} = \bar{C}_{ijmn}\bar{\varepsilon}_{mn} - \bar{e}_{nij}\bar{E}_n - \bar{q}_{nij}\bar{H}_n,$$

$$\bar{D}_i = \bar{e}_{imn}\bar{\varepsilon}_{mn} + \bar{\kappa}_{in}\bar{E}_n + \bar{\lambda}_{ni}\bar{H}_n$$

$$\bar{B}_i = \bar{q}_{imn}\bar{\varepsilon}_{mn} + \bar{\lambda}_{in}\bar{E}_n + \bar{\Gamma}_{in}H_n \tag{12a}$$

or, in the unified notation,

$$\bar{\Sigma}_{iJ} = \bar{L}_{iJMn}\bar{Z}_{Mn}, \tag{12b}$$

where the overbar denotes quantities associated with the entire composite.

3. Evalution of magneto–electro–elastic Eshelby tensors

The magneto–electro–elastic Eshelby tensors, S_{MnAb}, given by Eq. (7) readily demonstrate that

$$S_{MnAb} = \begin{cases} S_{mnab} = S_{nmab} = S_{nmba} = S_{mnba} & M,\ A \le 3, \\ S_{mn4b} = S_{nm4b} & M \le 3,\ A = 4, \\ S_{mn5b} = S_{nm5b} & M \le 3,\ A = 5, \\ S_{4nab} = S_{4nba} & M = 4,\ A \le 3, \\ S_{5nab} = S_{5nba} & M = 5,\ A \le 3. \end{cases} \tag{13}$$

With the help of the symmetry relations above, one can show that the number of non-zero Eshelby tensors is 41 for an ellipsoidal inclusion embedded in a transversely isotropic piezomagnetic medium, and among them only 28 are independent. Since the independent Eshelby tensors are of fundamental importance in the later evaluation of V^{-1}_{AbqR} and $\bar{L}_{iJMn}$, it is useful to list them herein:

$$S_{1111} = \frac{1}{4\pi}\left(C^0_{11}G_{1111} + C^0_{12}G_{1212} + C^0_{13}G_{1313} + q^0_{31}G_{1513}\right),$$

$$S_{1122} = \frac{1}{4\pi}\left(C^0_{12}G_{1111} + C^0_{11}G_{1212} + C^0_{13}G_{1313} + q^0_{31}G_{1513}\right),$$

$$S_{1133} = \frac{1}{4\pi}\left(C^0_{13}G_{1111} + C^0_{13}G_{1212} + C^0_{33}G_{1313} + q^0_{33}G_{1513}\right),$$

$$S_{1153} = \frac{1}{4\pi}\left(q^0_{31}G_{1111} + q^0_{31}G_{1212} + q^0_{33}G_{1313} - \Gamma^0_{33}G_{1513}\right),$$

$$S_{2211} = \frac{1}{4\pi}\left(C^0_{11}G_{1212} + C^0_{12}C_{12} + C^0_{13}G_{2323} + q^0_{31}G_{2532}\right),$$

$$S_{2222} = \frac{1}{4\pi}\left(C^0_{12}G_{1212} + C^0_{11}G_{2222} + C^0_{13}G_{2323} + q^0_{31}G_{2523}\right),$$

$$S_{2233} = \frac{1}{4\pi}\left(C^0_{13}G_{1212} + C^0_{13}G_{2222} + C^0_{33}G_{2323} + q^0_{33}G_{2523}\right),$$

$$S_{2253} = \frac{1}{4\pi}\left(q^0_{31}G_{1212} + q^0_{31}G_{2222} + q^0_{33}G_{2323} - \Gamma^0_{33}G_{2523}\right),$$

$$S_{3311} = \frac{1}{4\pi}\left(C^0_{11}G_{1313} + C^0_{12}G_{2323} + C^0_{13}G_{3333} + q^0_{31}G_{3533}\right),$$

$$S_{3322} = \frac{1}{4\pi}\left(C^0_{12}G_{1313} + C^0_{11}G_{2323} + C^0_{13}G_{3333} + q^0_{31}G_{3533}\right),$$

$$S_{3333} = \frac{1}{4\pi}\left(C^0_{13}G_{1313} + C^0_{13}G_{2323} + C^0_{33}G_{3333} + q^0_{33}G_{3533}\right),$$

$$S_{3353} = \frac{1}{4\pi}\left(q^0_{31}G_{1313} + q^0_{31}G_{2323} + q^0_{33}G_{3333} - \Gamma^0_{33}G_{3533}\right),$$

$$S_{2323} = \frac{1}{8\pi}\left\{C^0_{44}(G_{2233} + 2G_{2323} + G_{3322}) + q^0_{15}(G_{2523} + G_{3522})\right\},$$

$$S_{2352} = \frac{1}{8\pi}\left\{q^0_{15}(G_{2233} + 2G_{2323} + G_{3322}) - \Gamma^0_{11}(G_{2523} + G_{3522})\right\},$$

$$S_{1313} = \frac{1}{8\pi}\left\{C^0_{44}(G_{1133} + 2G_{1313} + G_{3311}) + q^0_{15}(G_{1513} + G_{3511})\right\},$$

$$S_{1351} = \frac{1}{8\pi}\left\{q^0_{15}(G_{1133} + 2G_{1313} + G_{3311}) - \Gamma^0_{11}(G_{1513} + G_{3511})\right\},$$

$$S_{1212} = \frac{1}{16\pi}\left(C^0_{11} - C^0_{12}\right)(G_{1122} + 2G_{1212} + G_{2211}),$$

$$S_{4141} = \frac{-1}{4\pi}\kappa^0_{11}G_{4411},$$

$$S_{4242} = \frac{-1}{4\pi}\kappa^0_{11}G_{4422},$$

$$S_{4343} = \frac{-1}{4\pi}\kappa^0_{33}G_{4433},$$

$$S_{5113} = \frac{1}{4\pi}\left(C^0_{44}G_{1513} + C^0_{44}G_{3511} + q^0_{15}G_{5511}\right),$$

$$S_{5151} = \frac{1}{4\pi}\left(q^0_{15}G_{1513} + q^0_{15}G_{3511} - \Gamma^0_{11}G_{5511}\right),$$

$$S_{5223} = \frac{1}{4\pi}\left(C^0_{44}G_{2523} + C^0_{44}G_{3522} + q^0_{15}G_{5522}\right),$$

$$S_{5252} = \frac{1}{4\pi}\left(q^0_{15}G_{2523} + q^0_{15}G_{3522} - \Gamma^0_{11}G_{5522}\right),$$

$$S_{5311} = \frac{1}{4\pi}\left(C^0_{11}G_{1513} + C^0_{12}G_{2523} + C^0_{13}G_{5333} + q^0_{31}G_{5533}\right),$$

$$S_{5322} = \frac{1}{4\pi}\left(C^0_{12}G_{1513} + C^0_{11}G_{2523} + C^0_{13}G_{3533} + q^0_{31}G_{5533}\right),$$

$$S_{5333} = \frac{1}{4\pi}\left(C^0_{13}G_{1513} + C^0_{13}G_{2523} + C^0_{33}G_{3533} + q^0_{33}G_{5533}\right),$$

$$S_{5543} = \frac{1}{4\pi}\left(q^0_{31}G_{1513} + q^0_{31}G_{2523} + q^0_{33}G_{5333} - \Gamma^0_{33}G_{5533}\right). \tag{14}$$

In derivation of the foregoing equations, the generalized Voigt two-index notation:

$$\begin{aligned} &11\to 1, \quad 22\to 2, \quad 33\to 3, \quad 23\to 4, \quad 31\to 5, \quad 12\to 6,\\ &41\to 7, \quad 42\to 8, \quad 43\to 9, \quad 51\to 10, \quad 52\to 11, \quad 53\to 12, \end{aligned} \tag{15}$$

has been adopted for the magneto–electro–elastic moduli L^0_{iJAb}.

It is observed that the tensors in Eq. (14) can be divided into three categories. The first category consists of those tensors related to the elastic response under eigenstrains, i.e. S_{1111}, S_{1122}, S_{1133}, S_{2211}, S_{2222}, S_{2233}, S_{1212}, S_{1313}, S_{2323}, S_{3311}, S_{3322} and S_{3333}. The second involves those related to the piezomagnetic response due to the initial piezomagnetic fields of the same kind, S_{4141}, S_{4242}, S_{4343}, S_{5151}, S_{5252} and S_{5353}. The third category includes elastic and magnetic interactive terms: S_{1153}, S_{2253}, S_{3353}, S_{1351}, S_{2352}, S_{5113}, S_{5223}, S_{5311}, S_{5322} and S_{5333}. Note that the tensors in the first and second categories are dimensionless, while the interactive terms in the third category relating dissimilar physical quantities are dimensional.

Next, we attempt to analytically explore the magneto–electro–elastic Eshelby tensors for some practical inclusions in the micromechanics and composite communities, such as elliptical cylinder, circular cylinder, penny shape and ribbon, embedded in transversely isotropic piezomagnetic materials. To this end, complete explicit expressions of the determinant $D(\bar{\xi})$ and the cofactor $N_{MJ}(\bar{\xi})$ of the matrix $[L^0_{iJMn}\bar{\xi}_i\bar{\xi}_n]$ are carried out first, followed by substituting $D(\bar{\xi})$ and $N_{MJ}(\bar{\xi})$ into G_{MJin}, given by Eq. (8). Consequently, complete explicit expressions for the corresponding G_{MJin} are then obtained. Having the explicit expressions of G_{MJin} in hand, closed-form expressions of the magneto–electro–elastic Eshelby tensors for piezomagnetic materials can be obtained below.

(a) Elliptical cylinder ($a_1/a_2 = a$, $a_3 \to \infty$):

$$S_{1111} = \frac{(2+3a)C^0_{11} + aC^0_{12}}{2(1+a)^2 C^0_{11}},$$

$$S_{2222} = \frac{(3a+2a^2)C^0_{11} + aC^0_{12}}{2(1+a)^2 C^0_{11}},$$

$$S_{1122} = \frac{-aC^0_{11} + (2+a)C^0_{12}}{2(1+a)^2 C^0_{11}},$$

$$S_{2211} = \frac{-aC^0_{11} + (a+2a^2)C^0_{12}}{2(1+a)^2 C^0_{11}},$$

$$S_{1133} = \frac{C^0_{13}}{(1+a)C^0_{11}},$$

$$S_{2233} = \frac{aC^0_{13}}{(1+a)C^0_{11}},$$

$$S_{1153} = \frac{q^0_{31}}{(1+a)C^0_{11}},$$

$$S_{2253} = \frac{aq^0_{31}}{(1+a)C^0_{11}},$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \frac{(1+a+a^2)C^0_{11} - aC^0_{12}}{2(1+a)^2C^0_{11}},$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = \frac{1}{2(1+a)},$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = \frac{a}{2(1+a)},$$

$$S_{4141} = S_{5151} = \frac{1}{1+a}$$

$$S_{4242} = S_{5252} = \frac{a}{1+a}. \tag{16}$$

(b) Circular cylinder ($a_1 = a_2$, $a_3 \rightarrow \infty$):

$$S_{1111} = S_{2222} = \frac{5C^0_{11} + C^0_{12}}{8C^0_{11}},$$

$$S_{1122} = S_{2211} = \frac{-C^0_{11} + 3C^0_{12}}{8C^0_{11}},$$

$$S_{1133} = S_{2233} = \frac{C^0_{13}}{2C^0_{11}},$$

$$S_{1153} = S_{2253} = \frac{q^0_{31}}{2C^0_{11}},$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \frac{3C^0_{11} - C^0_{12}}{8C^0_{11}},$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = S_{2323} = S_{2332} = S_{3223} = S_{3232} = \frac{1}{4}$$

$$S_{4141} = S_{4242} = S_{5151} = S_{5252} = \frac{1}{2}. \tag{17}$$

(c) Disk ($a_1 = a_2$, $a_3 \to 0$):

$$S_{1313} = S_{1331} = S_{3131} = S_{3113} = S_{2323} = S_{2332} = S_{3232} = S_{3223} = \frac{1}{2},$$

$$S_{1351} = S_{3151} = S_{2352} = S_{3252} = \frac{q_{15}^0}{2C_{44}^0},$$

$$S_{3311} = S_{3322} = \frac{q_{31}^0 q_{33}^0 + C_{13}^0 \Gamma_{33}^0}{q_{33}^{0^2} + C_{33}^0 \Gamma_{33}^0},$$

$$S_{5311} = S_{5322} = \frac{C_{13}^0 q_{33}^0 - C_{33}^0 q_{31}^0}{2\left(q_{33}^{0^2} + C_{33}^0 \Gamma_{33}^0\right)}$$

$$S_{3333} = S_{4343} = S_{5353} = 1. \tag{18}$$

(d) Ribbon ($a_1 \ll a_2$, $a_1/a_1 = a$, $a_3 \to \infty$):

$$S_{1111} = 1 - \frac{a\left(C_{11}^0 - C_{12}^0\right)}{2C_{11}^0},$$

$$S_{1122} = -\frac{a}{2} + \frac{(2 - 3a)C_{12}^0}{2C_{11}^0},$$

$$S_{2211} = \frac{a\left(C_{12}^0 - C_{11}^0\right)}{2C_{11}^0},$$

$$S_{2222} = \frac{a\left(3C_{11}^0 + C_{12}^0\right)}{2C_{11}^0},$$

$$S_{1133} = \frac{(1 - a)C_{13}^0}{C_{11}^0},$$

$$S_{2233} = \frac{aC^0_{13}}{C^0_{11}},$$

$$S_{1153} = \frac{(1-a)q^0_{31}}{C^0_{11}},$$

$$S_{2253} = \frac{aq^0_{31}}{C^0_{11}},$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \frac{1}{2} - \frac{a\left(C^0_{11} + C^0_{12}\right)}{2C^0_{11}},$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = \frac{1-a}{2},$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = \frac{a}{2},$$

$$S_{4141} = S_{5151} = 1 - a$$

$$S_{4242} = S_{5252} = a. \tag{19}$$

4. Closed-form solutions for the magnetoelectric coupling coefficients

With the explicit model for the effective magneto–electro–elastic moduli proposed in Section 2 and the analytical expressions for the magneto–electro–elastic Eshelby tensors in the preceding section, the magnetoelectric coupling effect of piezoelectric–piezomagnetic composites is to be investigated analytically in this section. It is readily shown from Eq. (9) with the aid of Eq. (11) that

$$\bar{\lambda}_{11} = -f\left\{2\kappa^0_{11}q^0_{15}V^{-1}_{4113} + \left(\Gamma^1_{11} - \Gamma^0_{11}\right)V^{-1}_{4115}\right\},$$

$$\bar{\lambda}_{22} = -f\left\{2\kappa^0_{11}q^0_{15}V^{-1}_{4223} + \left(\Gamma^1_{11} - \Gamma^0_{11}\right)V^{-1}_{4225}\right\}$$

$$\bar{\lambda}_{33} = -f\left\{\kappa^0_{33}q^0_{31}\left(V^{-1}_{4311} + V^{-1}_{4322}\right) + \kappa^0_{33}q^0_{33}V^{-1}_{4333} + \left(\Gamma^1_{33} - \Gamma^0_{33}\right)V^{-1}_{4335}\right\}, \tag{20}$$

and $\bar{\lambda}_{ij} = 0$ otherwise. By inspection of the equations above it is seen that to analytically evaluate the magnetoelectric coupling coefficients, the inversion of the fourth-order tensor has to be carried out before proceeding any further. Therefore, a special scheme for the fourth-order tensor inversion must be developed which is outlined here. First, with the generalized Voigt two-index notation given by Eq. (15), a 12×12 matrix is constructed for a given fourth-order tensor. For example, the fourth-order tensor, V_{iJAb}, given by Eq. (10) for an ellipsoidal inclusion embedded in transversely isotropic piezomagnetic media, can be mapped into the following matrix:

$$\begin{bmatrix} V_{1111} & V_{1122} & V_{1133} & 0 & 0 & 0 & 0 & 0 & V_{1143} & 0 & 0 & V_{1153} \\ V_{2211} & V_{2222} & V_{2233} & 0 & 0 & 0 & 0 & 0 & V_{2243} & 0 & 0 & V_{2253} \\ V_{3311} & V_{3322} & V_{3333} & 0 & 0 & 0 & 0 & 0 & V_{3343} & 0 & 0 & V_{3353} \\ 0 & 0 & 0 & V_{2323} & 0 & 0 & 0 & V_{2342} & 0 & 0 & V_{2352} & 0 \\ 0 & 0 & 0 & 0 & V_{1313} & 0 & V_{1341} & 0 & 0 & V_{1351} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V_{1212} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V_{1413} & 0 & V_{1441} & 0 & 0 & V_{1451} & 0 & 0 \\ 0 & 0 & 0 & V_{2423} & 0 & 0 & 0 & V_{2442} & 0 & 0 & V_{2452} & 0 \\ V_{3411} & V_{3422} & V_{3433} & 0 & 0 & 0 & 0 & 0 & V_{3443} & 0 & 0 & V_{3453} \\ 0 & 0 & 0 & 0 & V_{1513} & 0 & 0 & 0 & 0 & V_{1551} & 0 & 0 \\ 0 & 0 & 0 & V_{2523} & 0 & 0 & 0 & 0 & 0 & 0 & V_{2552} & 0 \\ V_{3511} & V_{3522} & V_{3533} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & V_{3553} \end{bmatrix}. \qquad (21)$$

The non-zero entries of the above matrix are obtained directly by substituting Eq. (14) into Eq. (10) and by use of Eq. (13). Complete explicit expressions for these non-zero entries have been accomplished in this work; however, the results are too lengthy to be listed here. Shown in Appendix A are V_{iJAb} for elliptic cylindrical, circular cylindrical, penny-shaped, and ribbon-like inclusions. It should be noted that in mapping a tensor into a matrix through the generalized Voigt two-index notation, care should be taken in accounting for the shear strain terms, i.e. the factor of two. Thus the element in columns 4 to 6 of the matrix in Eq. (21) is two times their corresponding tensor component.

Next, the 12×12 matrix is inverted and is used to map the corresponding inverse tensor as

$$\begin{bmatrix} V^{-1}_{1111} & V^{-1}_{1122} & V^{-1}_{1133} & 0 & 0 & 0 & 0 & 0 & V^{-1}_{1134} & 0 & 0 & V^{-1}_{1135} \\ V^{-1}_{2211} & V^{-1}_{2222} & V^{-1}_{2233} & 0 & 0 & 0 & 0 & 0 & V^{-1}_{2234} & 0 & 0 & V^{-1}_{2235} \\ V^{-1}_{3311} & V^{-1}_{3322} & V^{-1}_{3333} & 0 & 0 & 0 & 0 & 0 & V^{-1}_{3334} & 0 & 0 & V^{-1}_{3335} \\ 0 & 0 & 0 & V^{-1}_{2323} & 0 & 0 & 0 & V^{-1}_{2324} & 0 & 0 & V^{-1}_{2325} & 0 \\ 0 & 0 & 0 & 0 & V^{-1}_{1313} & 0 & V^{-1}_{1314} & 0 & 0 & V^{-1}_{1315} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & V^{-1}_{1212} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & V^{-1}_{4113} & 0 & V^{-1}_{4114} & 0 & 0 & V^{-1}_{4115} & 0 & 0 \\ 0 & 0 & 0 & V^{-1}_{4223} & 0 & 0 & 0 & V^{-1}_{4224} & 0 & 0 & V^{-1}_{4225} & 0 \\ V^{-1}_{4311} & V^{-1}_{4322} & V^{-1}_{4333} & 0 & 0 & 0 & 0 & 0 & V^{-1}_{4334} & 0 & 0 & V^{-1}_{4335} \\ 0 & 0 & 0 & 0 & V^{-1}_{5113} & 0 & V^{-1}_{5114} & 0 & 0 & V^{-1}_{5115} & 0 & 0 \\ 0 & 0 & 0 & V^{-1}_{5223} & 0 & 0 & 0 & V^{-1}_{5113} & 0 & 0 & V^{-1}_{5225} & 0 \\ V^{-1}_{5311} & V^{-1}_{5322} & V^{-1}_{5333} & 0 & 0 & 0 & 0 & 0 & V^{-1}_{5334} & 0 & 0 & V^{-1}_{5335} \end{bmatrix}. \qquad (22)$$

As in mapping a tensor into a matrix, in going from the inverse matrix back to the corresponding tensor, each element in columns 4 to 6 of the preceding matrix is divided by 2 to obtain the corresponding inverse tensor component. With this scheme, the evaluation of V^{-1}_{AbiJ} is then completed. Explicit expressions for the components of V^{-1}_{AbiJ} appearing in Eq. (19) are tabulated in Appendix B.

Once the inverse of the tensor V_{iJAb} is obtained, it can be used with Eq. (19) and Eqs. (B1), (B2) and (B3) to obtain closed-form solutions of the magnetoelectric coupling for elliptic cylindrical, circular cylindrical, penny-shaped, and ribbon-like inclusions. After some straightforward but tedious algebraic manipulations, the closed-form solutions are written out compactly as follows.

(a) Elliptical cylinder ($a_1/a_2 = a$, $a_3 \to \infty$):

$$\bar{\lambda}_{11} = \frac{-(1+a)^2 f(1-f)(1-a)e^1_{15}\kappa^0_{11}q^0_{15}\Gamma^1_{11}}{(a+f)^2 q^{02}_{15} Y_{11} + (1-f)^2 e^{12}_{15} Y_{12} + Y_{11}Y_{12}Y_{13}},$$

$$\bar{\lambda}_{22} = \frac{-a(1+a)^2 f(1-f)(1-a)e^1_{15}\kappa^0_{11}q^0_{15}\Gamma^1_{11}}{a^2 e^{12}_{15}(1-f)^2 Y_{21} + (1+af)^2 q^{02}_{15} Y_{22} + Y_{21}Y_{22}Y_{23}}$$

$$\bar{\lambda}_{33} = \frac{-2f(1-f)q^0_{31}e^1_{31}Y_{30}}{2(1+a^2)Y_{31} + a(1-f)^2 Y_{32} + aY_{33}}. \tag{23}$$

(b) Circular cylinder ($a_1 = a_2$, $a_3 \to \infty$):

$$\bar{\lambda}_{11} = \bar{\lambda}_{22} = \frac{-4f(1-f)e^1_{15}q^0_{15}\kappa^0_{11}\Gamma}{(1+f)\kappa^0_{11}Y_{11} + (1-f)\left(\kappa^1_{11}Y_{11} + Y_{12}\right)}$$

$$\bar{\lambda}_{33} = \frac{2f(1-f)e^1_{31}q^0_{31}}{(1-f)\left(C^0_{12} - C^1_{12} - C^1_{11}\right) - (1+f)C^0_{11}}. \tag{24}$$

(c) Disk ($a_1 = a_2$, $a_3 \to 0$):

$$\bar{\lambda}_{11} = \bar{\lambda}_{22} = \frac{-f(1-f)e^1_{15}q^0_{15}}{fC^0_{44} + (1-f)C^1_{44}}$$

$$\bar{\lambda}_{33} = \frac{-f(1-f)e^1_{33}\kappa^0_{33}q^0_{33}\Gamma^1_{33}Y_{30}}{Y_{30}Y_{31} + (1-f)Y_{32}}. \tag{25}$$

(d) Ribbon ($a_1 \ll a_2$, $a_1/a_2 = a$, $a_3 \to \infty$):

$$\bar{\lambda}_{11} = \frac{(1-a)f(1-f)e^1_{15}\kappa^0_{11}q^0_{15}\Gamma^1_{11}}{(a+f-af)^2 q^{02}_{15}Y_{11} + (1-a)^2(1-f)^2 e^{12}_{15}Y_{12} - Y_{11}Y_{12}Y_{13}},$$

$$\bar{\lambda}_{22} = \frac{af(1-f)(1-a)e^1_{15}\kappa^0_{11}q^0_{15}\Gamma^1_{11}}{(1-a+af)\left(Y_{22} - Y_{21}\Gamma^0_{11}\right) - a(1-f)\Gamma^1_{11}Y_{21}}$$

$$\bar{\lambda}_{33} = \frac{-f(1-f)fq^0_{31}e^1_{31}Y_{30}}{2a^2(1-f)^2 Y_{31} + Y_{32} + a(1-f)Y_{33}}. \tag{26}$$

The coefficients Y_{11}–Y_{33} in Eqs. (23)–(26) are listed in Appendix C.

It is seen from the above equations that the magnetoelectric coupling is a function of phase properties, volume fraction, and inclusion shape. It is also of interest to examine the behavior of the above solutions for the two-phase piezoelectric–piezomagnetic composite in the low and high concentration limits. As $f \to 0$ and $f \to 1$, the magnetoelectric coupling coefficients vanish. This verifies that the magnetoelectric coupling coefficients are absent in each constituent.

5. Concluding remarks

The most significant work to follow the investigation presented in this paper is the deveopment of an analytical prediction for the magnetoelectric coupling coefficients of a piezoelectric–piezomagnetic composite, as shown in Eq. (20). The ingredients for such a task are completely contained in the present article. Another valuable result is the closed-form expressions for a set of nine tensors for four practical inclusions in the micromechanics and composite communities: elliptical cylinder, circular cylinder, penny shape, and ribbon. These tensors are referred to as the magneto–electro–elastic Eshelby tensors analogous to the Eshelby tensor in elasticity. With these tensors, the magnetoelectric coupling coefficients are then obtained in closed forms as given by Eqs. (23)–(26) for the mentioned four inclusions. These results could provide us with an insight into how a piezoelectric–piezomagnetic composite material consisting of inclusions will perform and would be helpful in understanding the magneto–electric–elastic behavior of the composite. Also, the method presented here can be equally applied to piezoelectric materials containing a finite concentration of piezomagnetic inclusions.

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Appendix A

The non-zero elements in Eq. (21) for elliptic cylindrical, circular cylindrical, penny-shaped, and ribbon-like inclusions are listed as follows.

(a) Elliptical cylinder ($a_1/a_2 = a$, $a_3 \to \infty$):

$$V_{1111} = C_{11}^{0} + \frac{(1-f)}{2(1+a)^2 C_{11}^{0}} \Big\{ C_{11}^{0}\big[(2+3a)C_{11}^{1} - aC_{12}^{1}\big] - (2+3a)C_{11}^{0^2} + aC_{12}^{0}\big(C_{11}^{1} + C_{12}^{1} + 2aC_{12}^{1} - (1+2a)C_{12}^{0}\big)\Big\},$$

$$V_{1122} = C_{12}^{0} + \frac{C_{11}^{0}(1-f)}{2(1+a)^2 C_{11}^{0}} \Big\{ 2C_{12}^{0}\big(C_{11}^{1} - C_{11}^{0}\big) + 2a^2 C_{11}^{0}\big(C_{12}^{1} - C_{12}^{0}\big) + a\Big[C_{11}^{0^2} - C_{11}^{0}\big(4C_{12}^{0} + C_{11}^{1} - 3C_{12}^{1}\big) + C_{12}^{0}\big(C_{11}^{1} + C_{12}^{1} - C_{12}^{0}\big)\Big]\Big\},$$

$$V_{1133} = C_{13}^{0} + \frac{C_{13}^{0}(1-f)\big\{-C_{11}^{0} + C_{11}^{1} + a\big(-C_{12}^{0} + C_{12}^{1}\big)\big\}}{(1+a)C_{11}^{0}},$$

$$V_{1212} = \frac{C_{11}^{0} - C_{12}^{0}}{2} - \frac{(1-f)\big[(1+a+a^2)C_{11}^{0} - aC_{12}^{0}\big]\big(C_{11}^{0} - C_{12}^{0} - C_{11}^{1} + C_{12}^{1}\big)}{2(1+a)^2 C_{11}^{0}},$$

$$V_{1313} = C^0_{44} + \frac{(1-f)\left(C^1_{44} - C^0_{44}\right)}{1+a},$$

$$V_{1413} = \frac{(1-f)e^1_{15}}{1+a},$$

$$V_{1513} = q^0_{15} - \frac{(1-f)q^0_{15}}{1+a},$$

$$V_{2211} = C^0_{12} + \frac{(1-f)}{2(1+a)^2 C^0_{11}}\Big\{2a^2 C^0_{12}\left(C^1_{11} - C^0_{11}\right) + 2C^0_{11}\left(C^1_{12} - C^0_{12}\right) + a\Big[C^{0^2}_{11} - C^0_{11}$$

$$\left(4C^0_{12} + C^1_{11} - 3C^1_{12}\right) + C^0_{12}\left(C^1_{11} + C^1_{12} - C^0_{12}\right)\Big]\Big\},$$

$$V_{2222} = C^0_{11} + \frac{(1-f)}{2(1+a)^2 C^0_{11}}\Big\{-2a^2 C^0_{11}\left(C^0_{11} - C^1_{11}\right) + 2C^0_{12}\left(C^1_{12} - C^0_{12}\right) + a\Big[-3C^{0^2}_{11} + C^0_{11}$$

$$\left(3C^1_{11} - C^1_{12}\right) + C^0_{12}\left(C^1_{11} + C^1_{12} - C^0_{12}\right)\Big]\Big\},$$

$$V_{2233} = C^0_{13} + \frac{(1-f)C^0_{13}\left\{C^1_{12} - C^0_{12} + a\left(C^1_{11} - C^0_{11}\right)\right\}}{(1+a)C^0_{11}},$$

$$V_{2323} = C^0_{44} + \frac{a(1-f)\left(C^1_{44} - C^0_{44}\right)}{1+a},$$

$$V_{2423} = \frac{a(1-f)e^1_{15}}{1+a},$$

$$V_{2523} = q^0_{15} - \frac{a(1-f)q^0_{15}}{1+a},$$

$$V_{3311} = C^0_{13} - \frac{(1-f)\left(C^0_{11} + aC^0_{12}\right)\left(C^0_{13} - C^1_{13}\right)}{(1+a)C^0_{11}},$$

$$V_{3322} = C^0_{13} - \frac{(1-f)\left(aC^0_{11} + C^0_{12}\right)\left(C^0_{13} - C^1_{13}\right)}{(1+a)C^0_{11}},$$

$$V_{3411} = \frac{(1-f)\left(C^0_{11} + aC^0_{12}\right)e^1_{31}}{(1+a)C^0_{11}},$$

$$V_{3422} = \frac{(1-f)\left(aC^0_{11} + C^0_{12}\right)e^1_{31}}{(1+a)C^0_{11}},$$

$$V_{3433} = \frac{(1-f)C^0_{13}e^1_{31}}{C^0_{11}},$$

$$V_{3511} = q^0_{31} - \frac{(1-f)\left(C^0_{11} + aC^0_{12}\right)q^0_{31}}{(1+a)C^0_{11}},$$

$$V_{3522} = q^0_{31} - \frac{(1-f)\left(aC^0_{11} + C^0_{12}\right)q^0_{31}}{(1+a)C^0_{11}},$$

$$V_{3533} = q^0_{33} - \frac{(1-f)C^0_{13}q^0_{31}}{C^0_{11}},$$

$$V_{1341} = \frac{(1-f)e^1_{15}}{1+a},$$

$$V_{1441} = -\kappa^0_{11} + \frac{(1-f)\left(\kappa^0_{11} - \kappa^1_{11}\right)}{1+a},$$

$$V_{2442} = -\kappa^0_{11} + \frac{a(1-f)\left(\kappa^0_{11} - \kappa^1_{11}\right)}{1+a},$$

$$V_{3443} = -\kappa^0_{33},$$

$$V_{1153} = q^0_{31} + \frac{(1-f)q^0_{31}\left[C^1_{11} - C^0_{11} + a\left(C^1_{12} - C^0_{12}\right)\right]}{(1+a)C^0_{11}},$$

$$V_{1351} = q^0_{15} - \frac{(1-f)q^0_{15}}{1+a},$$

$$V_{1551} = -\Gamma^0_{11} + \frac{(1-f)\left(\Gamma^0_{11} - \Gamma^1_{11}\right)}{1+a},$$

$$V_{2253} = q^0_{31} + \frac{(1-f)q^0_{31}\left[C^1_{12} - C^0_{12} + a\left(C^1_{11} - C^0_{11}\right)\right]}{(1+a)C^0_{11}},$$

$$V_{2352} = q^0_{15} - \frac{a(1-f)q^0_{15}}{1+a},$$

$$V_{2552} = -\Gamma^0_{11} + \frac{a(1-f)\left(\Gamma^0_{11} - \Gamma^1_{11}\right)}{1+a},$$

$$V_{3353} = q^0_{33} + \frac{(1-f)q^0_{31}\left(C^1_{13} - C^0_{13}\right)}{C^0_{11}},$$

$$V_{3453} = \frac{(1-f)e^1_{31}q^0_{31}}{C^0_{11}}$$

$$V_{3553} = -\Gamma^0_{33} - \frac{(1-f)q^{0^2}_{31}}{C^0_{11}}. \tag{A1}$$

(b) Circular cylinder ($a_1 = a_2$, $a_3 \to \infty$):

$$V_{1111} = C^0_{11} + \frac{(1-f)}{8C^0_{11}}\left\{-5C^{0^2}_{11} + C^0_{11}\left(5C^1_{11} - C^1_{12}\right) + C^0_{12}\left(3C^1_{12} - 3C^0_{12} + C^1_{11}\right)\right\},$$

$$V_{1122} = C^0_{12} + \frac{(1-f)}{8C^0_{11}}\left\{C^{0^2}_{11} - C^0_{11}\left(8C^0_{12} + C^1_{11} - 5C^1_{12}\right) + C^0_{12}\left(C^1_{12} - C^0_{12} + 3C^1_{11}\right)\right\},$$

$$V_{1133} = C^0_{13} + \frac{(1-f)C^0_{13}\left(C^1_{11} - C^0_{11} + C^1_{12} - C^0_{12}\right)}{2C^0_{11}},$$

$$V_{1212} = \frac{C^0_{11} - C^0_{12}}{2} - \frac{(1-f)}{8C^0_{11}}\left(3C^0_{11} - C^0_{12}\right)\left(C^0_{11} - C^0_{12} - C^1_{11} + C^1_{12}\right),$$

$$V_{1513} = q^0_{15} - \frac{1}{2}(1-f)q^0_{15},$$

$$V_{2323} = C^0_{44} + \frac{1}{2}(1-f)\left(C^1_{44} - C^0_{44}\right),$$

$$V_{3311} = C^0_{13} - \frac{(1-f)\left(C^0_{11} + C^0_{12}\right)\left(C^0_{13} - C^1_{13}\right)}{2C^0_{11}},$$

$$V_{3333} = C^0_{33} + \frac{(1-f)C^0_{13}\left(-C^0_{13} + C^1_{13}\right)}{C^0_{11}},$$

$$V_{3411} = \frac{(1-f)\left(C^0_{11} + C^0_{12}\right)e^1_{31}}{2C^0_{11}},$$

$$V_{3433} = \frac{(1-f)C^{0}_{13}e^{1}_{31}}{C^{0}_{11}},$$

$$V_{3511} = q^{0}_{31} - \frac{(1-f)\left(C^{0}_{11} + C^{0}_{12}\right)q^{0}_{31}}{2C^{0}_{11}},$$

$$V_{3533} = q^{0}_{33} - \frac{C^{0}_{13}(1-f)q^{0}_{31}}{C^{0}_{11}},$$

$$V_{1441} = -\kappa^{0}_{11} + \frac{1}{2}(1-f)\left(\kappa^{0}_{11} - \kappa^{0}_{11}\right),$$

$$V_{2342} = \frac{1}{2}e^{1}_{15}(1-f),$$

$$V_{3443} = -\kappa^{0}_{33},$$

$$V_{1153} = q^{0}_{31} + \frac{(1-f)\left(C^{1}_{12} + C^{1}_{12} - C^{0}_{11} - C^{0}_{12}\right)q^{0}_{31}}{2C^{0}_{11}},$$

$$V_{1551} = -\Gamma^{0}_{11} + \frac{1}{2}(1-f)\left(\Gamma^{0}_{11} - \Gamma^{1}_{11}\right),$$

$$V_{2352} = q^{0}_{15} - \frac{1}{2}(1-f)q^{0}_{15},$$

$$V_{3353} = q^{0}_{33} + \frac{(1-f)\left(C^{1}_{13} - C^{0}_{13}\right)q^{0}_{31}}{C^{0}_{11}},$$

$$V_{3453} = \frac{(1-f)e^{1}_{31}q^{0}_{31}}{C^{0}_{11}}$$

$$V_{3553} = -\Gamma^{0}_{33} - \frac{(1-f)q^{0^2}_{31}}{C^{0}_{11}}. \qquad \text{(A2)}$$

(c) Disk ($a_1 = a_2$, $a_3 \to 0$):

$$V_{1111} = C^{0}_{11} + \frac{(1-f)\left(C^{0}_{33}q^{0^2}_{31} - 2C^{0}_{13}q^{0}_{31}q^{0}_{33} + C^{1}_{13}q^{0}_{31}q^{0}_{33} - C^{0^2}_{13}\Gamma^{0}_{33} + C^{0}_{13}C^{1}_{13}\Gamma^{0}_{33}\right)}{q^{0^2}_{33} + C^{0}_{33}\Gamma^{0}_{33}},$$

$$V_{1122} = C_{12}^{0} + \frac{(1-f)\left(C_{33}^{0}q_{31}^{0^2} - 2C_{13}^{0}q_{31}^{0}q_{33}^{0} + C_{13}^{1}q_{31}^{0}q_{33}^{0} - C_{13}^{0^2}\Gamma_{33}^{0} + C_{13}^{0}C_{13}^{1}\Gamma_{33}^{0}\right)}{q_{33}^{0^2} + C_{33}^{0}\Gamma_{33}^{0}},$$

$$V_{1133} = C_{13}^{0} + (1-f)\left(C_{13}^{1} - C_{13}^{0}\right),$$

$$V_{1212} = \frac{C_{11}^{0} - C_{12}^{0}}{2},$$

$$V_{1313} = C_{44}^{0} + (1-f)\left(C_{44}^{1} - C_{44}^{0}\right),$$

$$V_{1413} = (1-f)e_{15}^{1},$$

$$V_{1513} = q_{15}^{0} - (1-f)q_{15}^{0},$$

$$V_{2323} = C_{44}^{0} + (1-f)\left(C_{44}^{1} - C_{44}^{0}\right),$$

$$V_{2423} = (1-f)e_{15}^{1},$$

$$V_{2523} = fq_{15}^{0},$$

$$V_{3311} = C_{13}^{0} + \frac{(1-f)}{q_{33}^{0^2} + C_{33}^{0}\Gamma_{33}^{0}}\left\{C_{33}^{1}\left(q_{31}^{0}q_{33}^{0} + C_{13}^{0}\Gamma_{33}^{0}\right) - C_{13}^{0}\left(q_{33}^{0^2} + C_{33}^{0}\Gamma_{33}^{0}\right)\right\},$$

$$V_{3333} = C_{33}^{0} + (1-f)\left(C_{33}^{1} - C_{33}^{0}\right),$$

$$V_{3411} = \frac{(1-f)e_{33}^{1}\left(q_{31}^{0}q_{33}^{0} + C_{13}^{0}\Gamma_{33}^{0}\right)}{q_{33}^{0^2} + C_{33}^{0}\Gamma_{33}^{0}},$$

$$V_{3433} = (1-f)e_{33}^{1},$$

$$V_{3533} = fq_{33}^{0},$$

$$V_{1441} = -\kappa_{11}^{0},$$

$$V_{3443} = -\kappa_{33}^{0},$$

$$V_{3511} = q_{31}^{0} - \frac{(1-f)}{q_{33}^{0^2} + C_{33}^{0}\Gamma_{33}^{0}}\left\{q_{31}^{0}\left[q_{33}^{0^2} + C_{33}^{0}\left(\Gamma_{33}^{0} - \Gamma_{33}^{1}\right)\right] + C_{13}^{0}q_{33}^{0}\Gamma_{33}^{1}\right\},$$

$$V_{1153} = q^0_{31},$$

$$V_{1451} = \frac{(1-f)e^1_{15}q^0_{15}}{C^0_{44}},$$

$$V_{1551} = -\Gamma^0_{11} - \frac{(1-f)q^{0^2}_{15}}{C^0_{44}},$$

$$V_{2352} = q^0_{15} + (1-f)q^0_{15}\left(-1 + C^1_{44}/C^0_{44}\right),$$

$$V_{3353} = q^0_{33}$$

$$V_{3553} = -\Gamma^0_{33}. \tag{A3}$$

(d) Ribbon ($a_1 \ll a_2$, $a_1/a_2 = a$, $a_3 \to \infty$):

$$V_{1111} = C^0_{11} + \frac{(1-f)}{2C^0_{11}}\left\{(-2+a)C^{0^2}_{11} + aC^0_{12}\left(C^1_{11} + C^1_{12} - C^0_{12}\right) - C^0_{11}\left[(-2+a)C^1_{11} + aC^1_{12}\right]\right\},$$

$$V_{1122} = C^0_{12} + \frac{(1-f)}{2C^0_{11}}\left\{2C^0_{12}\left(C^1_{11} - C^0_{11}\right) + a\left[C^{0^2}_{11} - C^0_{11}\left(C^1_{11} - 3C^1_{12}\right) + C^0_{12}\left(C^1_{12} - C^0_{12} - 3C^1_{11}\right)\right]\right\},$$

$$V_{1133} = C^0_{13} + (1-f)\left\{C^0_{13}\left[(-1+a)C^0_{11} + C^1_{11} - a\left(C^0_{12} + C^1_{11} - C^1_{12}\right)\right]\right\}/C^0_{11},$$

$$V_{1212} = \frac{C^0_{11} - C^0_{12}}{2} + \frac{(1-f)}{2C^0_{11}}\left[(-1+a)C^0_{11} + aC^0_{12}\right]\left(C^0_{11} - C^0_{12} - C^1_{11} + C^1_{12}\right),$$

$$V_{1313} = C^0_{44} + (1-f)(1-a)\left(C^1_{44} - C^0_{44}\right),$$

$$V_{1413} = (1-f)\left(e^1_{15} - ae^1_{15}\right),$$

$$V_{1513} = q^0_{15} - (1-a)(1-f)q^0_{15},$$

$$V_{2211} = C^0_{12} + \frac{(1-f)}{2C^0_{11}}\left\{a\left(C^0_{11} - C^0_{12}\right)\left(C^0_{11} + C^0_{12} - C^1_{11} - C^1_{12}\right) + 2C^0_{11}\left(C^1_{12} - C^0_{12}\right)\right\},$$

$$V_{2222} = C^0_{11} + \frac{(1-f)}{2C^0_{11}}\Big\{a\Big[C^0_{12}\big(3C^0_{12} + C^1_{11} - 3C^1_{12}\big) - 3C^{0^2}_{11} + C^0_{11}\big(3C^1_{11} - C^1_{12}\big)\Big] + 2C^0_{12}$$
$$\big(C^1_{12} - C^0_{12}\big)\Big\},$$

$$V_{2233} = C^0_{13} + \frac{(1-f)}{C^0_{11}}C^0_{13}\big[C^1_{12} - C^0_{12} + a\big(-C^0_{11} + C^0_{12} + C^1_{11} - C^1_{12}\big)\big],$$

$$V_{2323} = C^0_{44} + a(1-f)\big(C^1_{44} - C^0_{44}\big),$$

$$V_{2423} = a(1-f)e^1_{15},$$

$$V_{2523} = q^0_{15} - a(1-f)q^0_{15},$$

$$V_{3311} = C^0_{13} + \frac{(1-f)\big[(-1+a)C^0_{11} - aC^0_{12}\big]\big(C^0_{13} - C^1_{13}\big)}{C^0_{11}},$$

$$V_{3322} = C^0_{13} - \frac{(1-f)\big[a\big(C^0_{11} - C^0_{12}\big) + C^0_{12}\big]\big(C^0_{13} - C^1_{13}\big)}{C^0_{11}},$$

$$V_{3333} = C^0_{33} + \frac{(1-f)C^0_{13}\big(C^1_{13} - C^0_{13}\big)}{C^0_{11}},$$

$$V_{3411} = (1-f)e^1_{31}\big(1 - a + aC^0_{12}/C^0_{11}\big),$$

$$V_{3422} = \frac{(1-f)\big[a\big(C^0_{11} - C^0_{12}\big) + C^0_{12}\big]e^1_{31}}{C^0_{11}},$$

$$V_{3433} = \frac{C^0_{13}e^1_{31}(1-f)}{C^0_{11}},$$

$$V_{3511} = q^0_{31} + (1-f)q^0_{31}\big(a - 1 - aC^0_{12}/C^0_{11}\big),$$

$$V_{3522} = q^0_{31} - \frac{(1-f)\big[a\big(C^0_{11} - C^0_{12}\big) + C^0_{12}\big]q^0_{31}}{C^0_{11}},$$

$$V_{3533} = q^0_{33} - \frac{(1-f)C^0_{13}q^0_{31}}{C^0_{11}},$$

$$V_{1341} = (1-f)\left(e^1_{15} - ae^1_{15}\right),$$

$$V_{1441} = -\kappa^0_{11} + (1-a)(1-f)\left(\kappa^0_{11} - \kappa^1_{11}\right),$$

$$V_{2342} = a(1-f)e^1_{15},$$

$$V_{2442} = -\kappa^0_{11} + a(1-f)\left(\kappa^0_{11} - \kappa^1_{11}\right),$$

$$V_{3443} = -\kappa^0_{33},$$

$$V_{1153} = q^0_{31} + \frac{(1-f)q^0_{31}}{C^0_{11}}\left\{C^1_{11} - (1-a)C^0_{11} - a\left(C^0_{12} + C^1_{11} - C^1_{12}\right)\right\},$$

$$V_{1351} = q^0_{15} - (1-a)(1-f)q^0_{15},$$

$$V_{1551} = -\Gamma^0_{11} + (1-a)(1-f)\left(\Gamma^0_{11} - \Gamma^1_{11}\right),$$

$$V_{2253} = q_{31} + \frac{(1-f)q^0_{31}}{C^0_{11}}\left\{C^1_{12} - C^0_{12} + a\left(C^1_{11} - C^1_{12} - C^0_{11} + C^0_{12}\right)\right\},$$

$$V_{2352} = q^0_{15} - a(1-f)q^0_{15},$$

$$V_{2552} = -\Gamma^0_{11} + a(1-f)\left(\Gamma^0_{11} - \Gamma^1_{11}\right),$$

$$V_{3353} = q^0_{33} + \frac{(1-f)\left(C^1_{13} - C^0_{13}\right)q^0_{31}}{C^0_{11}},$$

$$V_{3453} = \frac{(1-f)e^1_{31}q^0_{31}}{C^0_{11}}$$

$$V_{3553} = -\Gamma^0_{33} - \frac{(1-f)q^{0^2}_{31}}{C^0_{11}}. \qquad \text{(A4)}$$

Appendix B

Explicit expressions for the components of V^{-1}_{AbiJ} appearing in Eq. (19) are given by

(a) Elliptical cylinder ($a_1/a_2 = a$, $a_3 \to \infty$):

$$V^{-1}_{4113} = (1+a)e^1_{15}(1-f)\big(a\Gamma^0_{11} + f(\Gamma^0_{11} - \Gamma^1_{11}) + \Gamma^1_{11}\big)\Big/2\Big\{(a+f)^2\big[a\kappa^0_{11} + f(\kappa^0_{11} - \kappa^1_{11}) + \kappa^1_{11}\big]q^{0^2}_{15} + e^{1^2}_{15}(1-f)^2\big[a\Gamma^0_{11} + f(\Gamma^0_{11} - \Gamma^1_{11}) + \Gamma^1_{11}\big] + \big(aC^0_{44} + C^1_{44} + C^0_{44}f - C^1_{44}f\big)\big[a\kappa^0_{11} + f(\kappa^0_{11} - \kappa^1_{11}) + \kappa^1_{11}\big]\big[a\Gamma^0_{11} + f(\Gamma^0_{11} - \Gamma^1_{11}) + \Gamma^1_{11}\big]\Big\},$$

$$V^{-1}_{4115} = (1+a)(1-f)(a+f)e^1_{15}q^0_{15}\Big/\Big\{(a+f)^2\big[a\kappa^0_{11} + f(\kappa^0_{11} - \kappa^1_{11}) + \kappa^1_{11}\big]q^{0^2}_{15} + e^{1^2}_{15}(1-f)^2\big[a\Gamma^0_{11} + f(\Gamma^0_{11} - \Gamma^1_{11}) + \Gamma^1_{11}\big] + \big(aC^0_{44} + C^1_{44} + C^0_{44}f - C^1_{44}f\big)\big[a\kappa^0_{11} + f(\kappa^0_{11} - \kappa^1_{11}) + \kappa^1_{11}\big]\big[a\Gamma^0_{11} + f(\Gamma^0_{11} - \Gamma^1_{11}) + \Gamma^1_{11}\big]\Big\},$$

$$V^{-1}_{4223} = a(1+a)(1-f)e^1_{15}\big[(a+f)\Gamma^0_{11} + a(1-f)\Gamma^1_{11}\big]\Big/2\Big\{\big[\kappa^1_{11} + af\kappa^0_{11} + a(1-f)\kappa^1_{11}\big]\big(q^0_{15} + afq^0_{15}\big)^2 + a^2e^{1^2}_{15}(1-f)^2\big[\Gamma^0_{11} + af\Gamma^0_{11} + a(1-f)\Gamma^1_{11}\big] + \big[C^0_{44} + aC^1_{44}(1-f) + C^0_{44}f - aC^1_{44}f\big]\big[\kappa^0_{11} + af\kappa^0_{11} + a(1-f)\kappa^1_{11}\big]\big[\Gamma^0_{11} + af\Gamma^0_{11} + a(1-f)\Gamma^1_{11}\big]\Big\},$$

$$V^{-1}_{4225} = a(1+a)(1-f)e^1_{15}\big(q^0_{15} + afq^0_{15}\big)\Big/\Big\{\big[\kappa^1_{11} + af\kappa^0_{11} + a(1-f)\kappa^1_{11}\big]\big(q^0_{15} + afq^0_{15}\big)^2 + a^2e^{1^2}_{15}(1-f)^2\big[\Gamma^0_{11} + af\Gamma^0_{11} + a(1-f)\Gamma^1_{11}\big] + \big[C^0_{44} + aC^1_{44}(1-f) + C^0_{44}f - aC^1_{44}f\big]\big[\kappa^0_{11} + af\kappa^0_{11} + a(1-f)\kappa^1_{11}\big]\big[\Gamma^0_{11} + af\Gamma^0_{11} + a(1-f)\Gamma^1_{11}\big]\Big\},$$

$$V^{-1}_{4311} = (1-f)e^1_{31}\Big\{aC^0_{12}\big(C^0_{12} + C^1_{11} - C^1_{12}\big)(-1+f) + C^{0^2}_{11}[2 + a(-1+3f)] + C^0_{11}\big[3a\big(C^1_{11} - C^1_{12}\big)(1-f) + c^0_{12}(-2+2a-4af)\big]\Big\}\Big/\Big\{\Big(2C^0_{11}\big(C^0_{11} - C^0_{12}\big)\big(C^1_{12} + C^0_{11}f - C^1_{11}f\big) + 2a^2C^0_{11}\big(C^0_{11} - C^0_{12}\big)\big(C^1_{11} + C^0_{11}f - C^1_{11}f\big) + a\Big(C^0_{12}\Big(C^{0^2}_{12} - C^{1^2}_{11} - 2C^0_{12}C^1_{12} + C^{1^2}_{12}\Big)(1-f)^2 + C^0_{11}(1-f)\Big(-C^{0^2}_{12} + 3C^{1^2}_{11} + 4C^0_{12}C^1_{12} - 3C^{1^2}_{12} + 3C^{0^2}_{12}f - 2C^0_{12}C^1_{11}f - 3C^{1^2}_{11}f - 6C^0_{12}C^1_{12}f + 3C^{1^2}_{12}f\Big) + C^{1^3}_{11}(1+3f^2) - C^{1^2}_{11}\big(C^0_{12} + 2C^1_{12} + 2C^1_{12}f - 6C^1_{11}f - 2C^2_{12}f + C^0_{12}f^2 + 6C^1_{11}f^2\big)\Big)\Big)\kappa^0_{33}\Big\},$$

$$V^{-1}_{4322} = (1-f)e^{1}_{31}\Big\{C^{0^2}_{11}(2a+3f-1) - (1-f)C^{0}_{12}\big(C^{0}_{12}+C^{1}_{11}-C^{1}_{12}\big) + C^{0}_{11}\big[3(1-f)\big(C^{1}_{11} - C^{1}_{12}\big) - 2C^{0}_{12}(a+2f-1)\big]\Big\}\Big/\kappa^{0}_{33}\Big\{2C^{0}_{11}\big(C^{0}_{11}-C^{0}_{12}\big)\big(C^{1}_{12}+C^{0}_{11}f-C^{1}_{11}f\big) + 2a^2C^{0}_{11}\big(C^{0}_{11} - C^{0}_{12}\big)\big(C^{1}_{11}+C^{0}_{11}f-C^{1}_{11}f\big) + a\Big[C^{0}_{12}\Big(C^{0^2}_{12}-C^{1^2}_{11}-2C^{0}_{12}C^{1}_{12}+C^{1^2}_{12}\Big)(1-f)^2 + C^{0}_{11}(1 - f)\Big(4C^{0}_{12}C^{1}_{12}-C^{0^2}_{12}+3C^{1^2}_{11}-3C^{1^2}_{12}+3C^{0^2}_{12}f-2C^{0}_{12}C^{1}_{11}f-3C^{1^2}_{11}f-6C^{0}_{12}C^{1}_{12}f + 3C^{1^2}_{12}f\Big) + C^{1^3}_{11}(1+3f^2) - C^{1^2}_{11}\big(C^{0}_{12}+2C^{1}_{12}+2C^{1}_{12}f-6C^{1}_{11}f-2C^{2}_{12}f+C^{0}_{12}f^2 + 6C^{1}_{11}f^2\big)\Big]\Big\}$$

$$V^{-1}_{4333} = V^{-1}_{4335} = 0. \tag{B1}$$

(b) Circular cylinder ($a_1 = a_2$, $a_3 \to \infty$):

$$V^{-1}_{4113} = V^{-1}_{4223} = (1-f)e^{1}_{15}\big(\Gamma^{0}_{11}+f\Gamma^{0}_{11}+\Gamma^{1}_{11}-f\Gamma^{1}_{11}\big)\Big/\Big\{\big(\kappa^{0}_{11}+f\kappa^{0}_{11}+\kappa^{1}_{11}-f\kappa^{1}_{11}\big)\big(q^{0}_{15} + fq^{0}_{15}\big)^2 + e^{1^2}_{15}(1-f)^2\big(\Gamma^{0}_{11}+f\Gamma^{0}_{11}+\Gamma^{1}_{11}-f\Gamma^{1}_{11}\big) + \big(C^{0}_{44}+C^{1}_{44}+C^{0}_{44}f+C^{1}_{44}f\big)\big(\kappa^{0}_{11} + f\kappa^{0}_{11}+\kappa^{1}_{11}-f\kappa^{1}_{11}\big)\big(\Gamma^{0}_{11}+f\Gamma^{0}_{11}+\Gamma^{1}_{11}-f\Gamma^{1}_{11}\big)\Big\},$$

$$V^{-1}_{4115} = V^{-1}_{4225} = -2(1-f)(1+f)e^{1}_{15}q_{15}\Big/\Big\{\big(\kappa^{0}_{11}+f\kappa^{0}_{11}+\kappa^{1}_{11}-f\kappa^{1}_{11}\big)\big(q^{0}_{15}+fq^{0}_{15}\big)^2 + e^{1^2}_{15}(1-f)^2\big(\Gamma^{0}_{11} + f\Gamma^{0}_{11}+\Gamma^{1}_{11}-\Gamma^{1}_{11}\big) + \big(C^{0}_{44}+C^{1}_{44}+C^{0}_{44}f-C^{1}_{44}f\big)\big(\kappa^{0}_{11}+f\kappa^{0}_{11}+\kappa^{1}_{11}-f\kappa^{1}_{11}\big)\big(\Gamma^{0}_{11} + f\Gamma^{0}_{11}+\Gamma^{1}_{11}-f\Gamma^{1}_{11}\big)\Big\},$$

$$V^{-1}_{4311} = V^{-1}_{4322} = \frac{(1-f)e^{1}_{31}}{\big\{(1-f)\big(C^{1}_{11}+C^{1}_{12}-C^{0}_{12}\big)+(1+f)C^{0}_{11}\big\}\kappa^{0}_{33}},$$

$$V^{-1}_{4333} = V^{-1}_{4335} = 0. \tag{B2}$$

(c) Disk ($a_1 = a_2$, $a_3 \to 0$):

$$V^{-1}_{4113} = V^{-1}_{4223} = \frac{(1-f)e^{1}_{15}}{2\big(C^{1}_{44}+C^{0}_{44}f-C^{1}_{44}f\big)\kappa^{0}_{11}},$$

$$V^{-1}_{4311} = V^{-1}_{4322} = (1-f)^2\big(C^0_{33}q^0_{31} - C^0_{13}q^0_{33}\big)e^1_{33}q^0_{33}\Gamma^1_{33} \Big/ \Big\{ -\kappa^0_{33}\Big(2C^0_{33}q^{0^2}_{31} - 4C^0_{13}q^0_{31}q^0_{33} + \big(C^0_{11} + C^0_{12}\big)q^{0^2}_{33} - 2C^{0^2}_{13}\Gamma^0_{33} + \big(C^0_{11} + C^0_{12}\big)C^0_{33}\Gamma^0_{33}\Big)\Big[C^1_{33}\Gamma^0_{33} + f\Big(q^{0^2}_{33} + \big(C^0_{33} - C^1_{33}\big)\Gamma^0_{33}\Big)\Big] + 2(-1+f)\kappa^0_{33}\big(C^0_{33}q^0_{31} - C^0_{13}q^0_{33}\big)\big[C^0_{33}fq^0_{31} + C^1_{33}\big(q^0_{31} - fq^0_{31}\big) + \big(C^1_{13}(-1+f) - C^0_{13}f\big)q^0_{33}\big]\Gamma^1_{33}\Big\},,$$

$$V^{-1}_{4333} = (1-f)e^1_{33}\Gamma^0_{33}\Big\{2C^0_{33}q^{0^2}_{31} - 4C^0_{13}q^0_{31}q^0_{33} + \big(C^0_{11} + C^0_{12}\big)q^{0^2}_{33} - 2C^{0^2}_{13}\Gamma^0_{33} + \big(C^0_{11} + C^0_{12}\big)C^0_{33}\Gamma^0_{33} + 2(1-f)q^0_{31}\big(C^0_{33}q^0_{31} - C^0_{13}q^0_{33}\big)\Gamma^1_{33}\Big\} \Big/ \Big\{\kappa^0_{33}\Big[2C^0_{33}q^{0^2}_{31} - 4C^0_{13}q^0_{31}q^0_{33} + \big(C^0_{11} + C^0_{12}\big)q^{0^2}_{33} - 2C^{0^2}_{13}\Gamma^0_{33} + \big(C^0_{11} + C^0_{12}\big)C^0_{33}\Gamma^0_{33}\Big]\Big[C^1_{33}\Gamma^0_{33} + f\Big(q^{0^2}_{33} + \big(C^0_{33} - C^1_{33}\big)\Gamma^0_{33}\Big)\Big] + 2(1-f)\kappa^0_{33}\big(C^0_{33}q^0_{31} - C^0_{13}q^0_{33}\big)\big[fC^0_{33}q^0_{31} + (1-f)C^1_{33}q^0_{31} - \big(C^1_{13}(1-f) + fC^0_{13}\big)q^0_{33}\big]\Gamma^1_{33}\Big\},$$

$$V^{-1}_{4335} = -e^1_{33}(1-f)q^0_{33}\Big\{2C^0_{33}q^{0^2}_{31} - 4C^0_{13}q^0_{31}q^0_{33} + \big(C^0_{11} + C^0_{12}\big)q^{0^2}_{33} - 2C^{0^2}_{13}\Gamma^0_{33} + \big(C^0_{11} + C^0_{12}\big)C^0_{33}\Gamma^0_{33}\Big\} \Big/ \Big\{\kappa^0_{33}\Big[2C^0_{33}q^{0^2}_{31} - 4C^0_{13}q^0_{31}q^0_{33} + \big(C^0_{11} + C^0_{12}\big)q^{0^2}_{33} - 2C^{0^2}_{13}\Gamma^0_{33} + \big(C^0_{11} + C^0_{12}\big)C^0_{33}\Gamma^0_{33}\Big]\Big[C^1_{33}\Gamma^0_{33} + f\Big(q^{0^2}_{33} + \big(C^0_{33} - C^1_{33}\big)\Gamma^0_{33}\Big)\Big] + 2(1-f)\kappa^0_{33}\big(C^0_{33}q^0_{31} - C^0_{13}q^0_{33}\big) \times \big[fC^0_{33}q^0_{31} + (1-f)C^1_{33}q^0_{31} - \big(C^1_{13}(1-f) + fC^0_{13}\big)q^0_{33}\big]\Gamma^1_{33}\Big\},$$

$$V^{-1}_{4115} = V^{-1}_{4225} = 0. \tag{B3}$$

(d) Ribbon ($a_1 \ll a_2$, $a_1/a_2 = a$, $a_3 \to \infty$):

$$V^{-1}_{4113} = (1-a)(1-f)e^1_{15}\big\{-\Gamma^0_{11} + (1-a)(1-f)\big(\Gamma^0_{11} - \Gamma^1_{11}\big)\big\} \Big/ \Big\{2(a+f-af)^2\big[-\kappa^0_{11} + (1-a)(1-f)\big(\kappa^0_{11} - \kappa^1_{11}\big)\big]q^{0^2}_{15} - (1-a)^2(1-f)^2e^{1^2}_{15}\big[-\Gamma^0_{11} + (1-a)(1-f)\big(\Gamma^0_{11} - \Gamma^1_{11}\big)\big] + \big[C^0_{44} - (1-a)(1-f)\big(C^0_{44} - C^1_{44}\big)\big]\big[-\kappa^0_{11} + (1-a)(1-f)\big(\kappa^0_{11} - \kappa^1_{11}\big)\big]\big[-\Gamma^0_{11} + (1-a)(1-f)\big(\Gamma^0_{11} - \Gamma^1_{11}\big)\big]\Big\},,$$

$$V^{-1}_{4115} = (1-a)(1-f)(a+f-af)e^1_{15}q^0_{15} \Big/ \Big\{ -(a+f-af)^2\big[-\kappa^0_{11} + (1-a)(1-f)\big(\kappa^0_{11} - \kappa^1_{11}\big)\big]q^{0^2}_{15} - (1-a)^2(1-f)^2 e^{1^2}_{15}\big[-\Gamma^0_{11} + (1-a)(1-f)\big(\Gamma^0_{11} - \Gamma^1_{11}\big)\big] + \big[C^0_{44} - (1-a)(1-f)\big(C^0_{44} - C^1_{44}\big)\big]\big[-\kappa^0_{11} + (1-a)(1-f)\big(\kappa^0_{11} - \kappa^1_{11}\big)\big]\big[-\Gamma^0_{11} + (1-a)(1-f)\big(\Gamma^0_{11} - \Gamma^1_{11}\big)\big]\Big\},$$

$$V_{4223} = a(1-f)e^1_{15}\big\{(a-1-af)\Gamma^0_{11} - a(1-f)\Gamma^1_{11}\big\} \Big/ \Big\{ -2[1-a(1-f)]^2\big[\kappa^0_{11} + a(1-f)\big(\kappa^1_{11} - \kappa^0_{11}\big)\big]q^{0^2}_{15} - 2a^2(1-f)^2 e^{1^2}_{15}\big[\Gamma^0_{11} + a(1-f)\big(\Gamma^1_{11} - \Gamma^0_{11}\big)\big] - 2\big[C^0_{44} + a\big(C^1_{44} - C^0_{44}\big)(1-f)\big] \times \big[\kappa^0_{11} + a(1-f)\big(\kappa^1_{11} - \kappa^0_{11}\big)\big]\big[\Gamma^0_{11} + a(1-f)\big(\Gamma^1_{11} - \Gamma^0_{11}\big)\big]\Big\},$$

$$V_{4225} = a(1-f)[1-a(1-f)]e^1_{15}q^0_{15} \Big/ \Big\{[1-a(1-f)]^2\big[\kappa^0_{11} + a(1-f)\big(\kappa^1_{11} - \kappa^0_{11}\big)\big]q^{0^2}_{15} - 2a^2(1-f)^2 e^{1^2}_{15}\big[\Gamma^0_{11} + a(1-f)\big(\Gamma^1_{11} - \Gamma^0_{11}\big)\big] - 2\big[C^0_{44} + a\big(C^1_{44} - C^0_{44}\big)(1-f)\big]\big[\kappa^0_{11} + a(1-f)\big(\kappa^1_{11} - \kappa^0_{11}\big)\big]\big[\Gamma^0_{11} + a(1-f)\big(\Gamma^1_{11} - \Gamma^0_{11}\big)\big]\Big\},$$

$$V^{-1}_{4311} = e^1_{31}(1-f)\Big\{C^{0^2}_{11}[-2 + a(5-3f) - 2a^2(1-f)] + a(1-2a)(1-f)C^0_{12}\big(C^0_{12} + C^1_{11} - C^1_{12}\big) + C^0_{11}\big[a(-3+2a)(1-f)\big(C^1_{11} - C^1_{12}\big) + 2(1-a)C^0_{12}(1-2a+2af)\big]\Big\} \Big/ \kappa^0_{33}\Big\{2a^2(1-f)^2 \times \big(C^0_{11} - C^0_{12}\big)\Big(C^{0^2}_{11} - C^{0^2}_{12} - 2C^0_{11}C^1_{11}C^{1^2}_{11} + 2C^0_{12}C^1_{12} - C^{1^2}_{12}\Big) - 2C^0_{11}\big(C^0_{11} - C^0_{12}\big)\big(C^1_{11} + C^0_{11}f - C^1_{11}f\big) - a(1-f)^2\big(C^0_{11} - C^0_{12} - C^1_{11} + C^1_{12}\big)\Big[-C^0_{12}\big(C^0_{12} - C^1_{11} - C^1_{12}\big) + C^{0^2}_{11}(-1+3f) + C^0_{11}\big(3C^1_{11} + 3C^1_{12} + 2C^0_{12}f - 3C^1_{11}f - 3C^1_{12}f\big)\Big]\Big\},$$

$$V^{-1}_{4322} = a(1-f)e^1_{31}\Big\{C^{0^2}_{11}(1+2a(-1+f)-3f)+(1-2a)(1-f)C^0_{12}\big(C^0_{12} + C^1_{11} - C^1_{12}\big) + C^0_{11}\big[(-3+2a)(1-f)\big(C^1_{11}-C^1_{12}\big) + C^0_{12}(-2+4a+4f - 4af)\big]\Big\}\Big/\kappa^0_{33}\Big\{2a^2(1-f)^2\big(C^0_{11}-C^0_{12}\big)\Big(C^{0^2}_{11} - C^{0^2}_{12} - 2C^0_{11}C^1_{11} + C^{1^2}_{11} + 2C^0_{12}C^1_{12} - C^{1^2}_{12}\Big) - 2C^0_{11}\big(C^0_{11}-C^0_{12}\big)\big(C^1_{11}+C^0_{11}f - C^1_{11}f\big) - a(1-f)^2\big(C^0_{11}-C^0_{12}-C^1_{11}+C^1_{12}\big)\Big[-C^0_{12}\big(C^0_{12} - C^1_{11} - C^1_{12}\big) + C^{0^2}_{11}(-1+3f) + C^0_{11}\big(3C^1_{11}+3C^1_{12}+2C^0_{12}f - 3C^1_{11}f - 3C^1_{12}f\big)\Big]\Big\}$$

$$V^{-1}_{4333} = V^{-1}_{4335} = 0. \tag{B4}$$

Appendix C

The coefficients $Y_{11} \rightarrow Y_{33}$ in Eq. (23) are given by:

$$Y_{11} = (a+f)\kappa^0_{11} + \kappa^1_{11} - f\kappa^1_{11},$$

$$Y_{12} = (a+f)\Gamma^0_{11} + \Gamma^1_{11} - f\Gamma^1_{11},$$

$$Y_{13} = aC^0_{44} + C^1_{44} + fC^0_{44} - fC^1_{44},$$

$$Y_{21} = \Gamma^0_{11} + af\Gamma^0_{11} - a(-1+f)\Gamma^1_{11},$$

$$Y_{22} = \kappa^0_{11} + af\kappa^0_{11} - a(-1+f)\kappa^1_{11},$$

$$Y_{23} = C^0_{44} + aC^1_{44} + af\big(C^0_{44} - C^1_{44}\big),$$

$$Y_{30} = -aC^0_{12}\big(C^0_{12} + C^1_{11} - C^1_{12}\big)(1-f),$$

$$Y_{31} = C^0_{12}\big(C^0_{12} - C^1_{11} - C^1_{12}\big)\big(C^0_{12} - C^1_{11} - C^1_{12}\big),$$

$$Y_{32} = aC^0_{44} + C^1_{44} + fC^0_{44} - fC^1_{44}$$

$$Y_{33}=C_{11}^{0^3}(1+3f^2)+C_{11}^{0}(1-f)\Big(4C_{12}^{0}C_{12}^{1}+3(1-f)(C_{11}^{1}-C_{12}^{1})(C_{11}^{1}+C_{12}^{1})-2fC_{12}^{0}(C_{11}^{1}+3C_{12}^{1})-(1-3f)C_{12}^{0^2}\Big)-C_{11}^{0^2}\Big[(1+f)^2C_{12}^{0}+2(1+f)(C_{12}^{1}-3fC_{11}^{1})\Big]. \tag{C1}$$

The coefficients Y_{11} and Y_{12} in Eq. (24) are given by

$$Y_{11}=(1+f)^2q_{15}^{0^2}+\{(1+f)C_{44}^{0}+(1-f)C_{44}^{1}\}\{(1+f)\Gamma_{11}^{0}+(1-f)\Gamma_{11}^{1}\}$$

$$Y_{12}=e_{15}^{1^2}(1-f)\{(1+f)\Gamma_{11}^{0}+(1-f)\Gamma_{11}^{1}\}. \tag{C2}$$

The coefficients Y_{30}, Y_{31} and Y_{32} in Eq. (25) are given by

$$Y_{30}=2C_{33}^{0}q_{31}^{0^2}-4C_{13}^{0}q_{31}^{0}q_{33}^{0}+(C_{11}^{0}+C_{12}^{0})q_{33}^{0^2}-2C_{13}^{0^2}\Gamma_{33}^{0}(C_{11}^{0}+C_{12}^{0})C_{33}^{0}\Gamma_{33}^{0},$$

$$Y_{31}=\kappa_{33}^{0}\Big(C_{33}^{1}\Gamma_{33}^{0}q_{33}^{0}+f\Big(q_{33}^{0^2}+(C_{33}^{0}-C_{33}^{1})\Gamma_{33}^{0}\Big)\Big)$$

$$Y_{32}=2\kappa_{33}^{0}(C_{33}^{0}q_{31}^{0}-C_{13}^{0}q_{33}^{0})\{fC_{33}^{0}q_{31}^{0}+(1-f)C_{33}^{1}q_{31}^{0}-[fC_{13}^{0}+(1-f)C_{13}^{1}]q_{33}^{0}\}\Gamma_{33}^{1}. \tag{C3}$$

The coefficients Y_{31}, Y_{32} and Y_{33} in Eq. (26) are given by

$$Y_{31}=(C_{11}^{0}-C_{12}^{0})(C_{11}^{0}+C_{12}^{0}-C_{11}^{1}-C_{12}^{1})(C_{11}^{0}-C_{12}^{0}-C_{11}^{1}+C_{12}^{1}),$$

$$Y_{32}=-2C_{11}^{0}(C_{11}^{0}-C_{12}^{0})(C_{11}^{1}+fC_{11}^{1}-fC_{11}^{1})$$

$$Y_{33}=(C_{11}^{0}-C_{12}^{0}-C_{11}^{1}+C_{12}^{1})\Big\{(1-f)C_{12}^{0}(C_{12}^{0}-C_{11}^{1}-C_{12}^{1})-(1-3f)C_{11}^{0^2}+C_{11}^{0}[3C_{11}^{1}+3C_{12}^{1}+2fC_{12}^{0}-3f(C_{11}^{1}+C_{12}^{1})]\Big\}. \tag{C4}$$

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